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A NEW SEQUENCE OF INTEGRAL TESTS FOR THE CONVERGENCE AND DIVERGENCE OF INFINITE SERIES.*

BY RAYMOND W. BRINK.

Introduction. The convergence or divergence of a given series is determined by the behavior of its partial sum, the sum of the first n terms of the series. If this partial sum approaches a finite limit as n increases indefinitely, the series is said to converge; otherwise it diverges.

For certain series, as for example a geometric series, it is possible to find explicitly a simple analytic expression for a function that for successive positive integral values of the variable takes on the values of the corresponding partial sums. Whenever such a function is known, the series is known to converge if the function approaches a finite limit when the variable becomes infinite, and to diverge or oscillate otherwise.

Often when an ordinary examination of a given series $u_0 + u_1 + \cdots$, brings to light no such function $S(x)$ having the property that

$$S(n) = S_n = u_0 + u_1 + u_2 + \cdots + u_{n-1},$$

one can readily find a simple function $u(x)$ such that $u(n) = u_n$. And in other cases where no simple function $S(x)$ and no simple function $u(x)$ are apparent, it is possible to find a simple form for a function $r(x)$ having the property that $r(n) = r_n = u_{n+1}/u_n$, the general test-ratio of the series.

More generally, it is often possible to find some simple function $f(x)$ such that

$$f(n) = f_n = \varphi(u_n, u_{n+1}, \cdots, u_{n+k}),$$

a function of the n th term of the series and of a certain number of the following terms. Since $u_n = S_{n+1} - S_n$, we can write

$$f(n) = f_n = \psi(S_n, S_{n+1}, \cdots, S_{n+k+1}).$$

Then the equation

$$f(x) = \psi(S(x), S(x+1), \cdots, S(x+k))$$

is a difference equation in $S(x)$, which, if $S(x)$ is properly defined in some initial interval, defines the function $S(x)$ elsewhere so that $S(n) = S_n$. An examination of this function $S(x)$ is then often sufficient to establish the convergence or divergence of the series in question.

* The results of this paper were presented to the American Mathematical Society, April, 1916. Many of the proofs, however, have been worked out in their present form since that time.

By means of this principle the following sequence of integral tests were suggested. Other tests may be suggested in a similar way. In discovering the form of a test involving a certain relation between terms of a series, instead of setting up the difference equation for $S(x)$ it is often simpler to set up a difference equation for a function involved in an earlier test, and then make use of the form of the earlier test. Thus the form of the test of § 2 is obtained from that of § 1. And from the form of the former test may be obtained that of § 4. Whether the form is discovered in this way or by setting up the difference equation in $S(x)$, which may always be done, the difference equations will be replaced by differential equations whose solutions will serve equally well to suggest the tests. The validity and usefulness of the tests suggested in this way will be examined by independent methods.

Throughout the paper the symbol $\{u_n\}$ will be used to denote a sequence u_0, u_1, u_2 , etc., whose general term u_n is positive. Since a finite number of terms do not affect questions of convergence, zero will be used as the lower limit of summation or integration in stating theorems. It seems better to omit certain changes in phraseology that suggest themselves for cases in which the conditions of a theorem apply only for values of a variable from a certain number on.

1. **The Maclaurin-Cauchy Integral Test.** Suppose we are given the series

$$u_0 + u_1 + u_2 + \cdots.$$

If $u(x)$ is a function such that $u(n) = u_n$, the difference equation

$$(1) \quad S(x+1) - S(x) = u(x)$$

defines a function $S(x)$ such that

$$S(n) = S_n = u_0 + u_1 + \cdots + u_{n-1},$$

provided that $S(x)$ is properly defined in some initial interval. From equation (1), by the mean value theorem,

$$(2) \quad S'(\xi) = u(x)$$

where $x < \xi < x+1$, and

$$S'(x) = \frac{dS}{dx}.$$

We may reasonably hope that under suitable restrictions, $S'(\xi)$ may for our purposes be replaced by $S'(x)$ and that we may write, approximately,

$$S(x) = C + \int_0^x u(x)dx.$$

We may, then, hope that if certain restrictions are placed upon $u(x)$, $S(x)$

will approach a limit or will fail to do so, and the given series will therefore converge or diverge according as the integral

$$\int_0^{\infty} u(x) dx$$

converges or diverges. Thus in this first application of our method is suggested the Maclaurin-Cauchy test* which is essentially as follows:

Given a sequence $\{u_n\}$. Let $u(x)$ be a positive integrable function having the property that $u(n) = u_n$, and that either

(α) *a constant k exists such that when $|x - x'| \leq 1$,*

$$\frac{u(x)}{u(x')} < k,$$

or

(β) *a constant m exists, positive or zero, such that, when $x' \geq x + m$,*

$$u(x) \geq u(x').$$

A necessary and sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} u_n$ is the convergence of the integral $\int_0^{\infty} u(x) dx$. The method of establishing this test and its many applications are too well known to require consideration here.

This test may be extended to test series of positive and negative or complex terms, as follows.

THEOREM I. *Given a series*

$$u_0 + u_1 + u_2 + \dots$$

Let $u(x)$ be an integrable function of the real variable x such that

$$(1) \quad u(n) = u_n,$$

$$(2) \quad \lim_{x \rightarrow \infty} u(x) = 0,$$

$$(3) \quad |u(x) - u_n| \leq v_n, \quad 0 \leq x - n \leq 1,$$

the series $\sum_{n=\mu}^{\infty} v_n$ being a convergent series. A necessary and sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} u_n$ is the convergence of the integral $\int_0^{\infty} u(x) dx$.

Proof. If we write

$$S_n^{\mu} = S_n - S_{\mu} = u_{\mu} + u_{\mu+1} + \dots + u_{n-1},$$

* Maclaurin: *Treatise on Fluxions*, I, p. 289, 1742. Cauchy: *Exercices de Mathématiques*, vol. 2, p. 221, 1827. *Oeuvres complètes*, 2^e série, vol. VII, p. 267.

we have

$$\begin{aligned}
 \left| \int_{\mu}^n u(x) dx - S_{\mu}^n \right| &= \left| \int_{\mu}^{\mu+1} (u(x) - u_{\mu}) dx + \int_{\mu+1}^{\mu+2} (u(x) - u_{\mu+1}) dx \right. \\
 &\quad \left. + \cdots + \int_{n-1}^n (u(x) - u_{n-1}) dx \right| \\
 &\leq \int_{\mu}^{\mu+1} |u(x) - u_{\mu}| dx + \int_{\mu+1}^{\mu+2} |u(x) - u_{\mu+1}| dx \\
 &\quad + \cdots + \int_{n-1}^n |u(x) - u_{n-1}| dx \\
 &\leq v_{\mu} + v_{\mu+1} + \cdots + v_{n-1} \leq \sum_{n=\mu}^{\infty} v_n.
 \end{aligned}$$

Since $\lim_{x \rightarrow \infty} u(x) = 0$, $\lim_{x \rightarrow \infty} \int_n^x u(x) dx = 0$, if $n \leq x \leq n+1$. Therefore

$$\lim_{x \rightarrow \infty} \left| \int_{\mu}^x u(x) dx - S_{\mu}^n \right| \leq \sum_{n=\mu}^{\infty} v_n, \quad \text{where} \quad n \leq x \leq n+1.$$

This inequality establishes the theorem, since by increasing μ we can make $\sum_{n=\mu}^{\infty} v_n$ as small as we wish. Moreover the inequality provides us with limits on the value of the series if the series converges, or on its oscillation if it oscillates.

Hardy* gives the following theorem: *If (1) $u(x)$ possesses a continuous derivative $u'(x)$, (2) $\lim_{x \rightarrow \infty} u(x) = 0$ and (3) the integral $\int_0^{\infty} |u'(x)| dx$ converges, the series $\sum_{n=0}^{\infty} u(n)$ converges, oscillates, or diverges together with the integral $\int_0^{\infty} u(x) dx$.* As Hardy points out, this theorem includes a useful test due to Bromwich.† On the other hand, Hardy's test is contained in Theorem I above. For suppose that the integral $\int_0^{\infty} |u'(x)| dx$ converges. We have

$$|u(x) - u_n| = \left| \int_n^x u'(x) dx \right| \leq \int_n^{n+1} |u'(x)| dx, \quad 0 \leq x - n \leq 1.$$

Then since $\sum_{n=0}^{\infty} \int_n^{n+1} |u'(x)| dx$ converges, $v_n = \int_n^{n+1} |u'(x)| dx$ satisfies the conditions of Theorem I. Bromwich and Hardy give some interesting examples of series of complex terms tested by these tests. For most applications the tests of Hardy and Bromwich and that of Theorem I are equally useful.

* Hardy: Proceedings of the London Math. Society, 2d ser., vol. 9 (1910), p. 127.

† Bromwich: Proceedings of the London Math. Society, 2d ser., vol. 6 (1908), p. 329.

2. Integral Tests of the Second Kind. Given a sequence $\{u_n\}$. Let $r_n = u_{n+1}/u_n$ be the *ratio* of the series $\sum_{n=0}^{\infty} u_n$. Then

$$\log r_n = \log u_{n+1} - \log u_n.$$

If $r(x)$ is a positive continuous function such that $r(n) = r_n$ and if $u(x)$ is properly defined in some initial interval, the difference equation

$$\log r(x) = \log u(x+1) - \log u(x)$$

defines a function $u(x)$ elsewhere so that $u(n) = u_n$. By the mean-value theorem

$$\log r(x) = \frac{u'(\xi)}{u(\xi)}$$

where $x < \xi < x+1$. We may reasonably hope that under suitable restrictions on $r(x)$, $u'(\xi)/u(\xi)$ will differ from $u'(x)/u(x)$ by magnitudes of a low order for large values of x , and that without excessive error we may write

$$u(x) = ke^{\int_0^x \log r(x) dx}$$

Using this form for $u(x)$ in the Maclaurin-Cauchy test, we obtain the integral

$$\int_0^{\infty} e^{\int_0^x \log r(x) dx} dx$$

as a form which under certain conditions may be expected to converge or diverge with the given series. In obtaining this form directly from the difference equation in $S(x)$ we are led to the differential equation

$$\frac{S''(x)}{S'(x)} = \log r(x),$$

whose solution is

$$S(x) = c + k \int_0^x e^{\int_0^x \log r(x) dx} dx.$$

The tests suggested in this way are much the most important of the present sequence of tests except for the Maclaurin-Cauchy test. These tests and others arising from them, as well as certain applications, have been treated by the writer in another paper where they were called *integral tests of the second kind*.* The three following theorems are quoted from that article.

THEOREM II. *Given the sequence $\{u_n\}$. Let $r_n = u_{n+1}/u_n$, and let $r(x)$ be a positive, integrable function having the property that $r(n) = r_n$, and that a*

* Transactions of the American Math. Society, vol. 19 (1918), p. 186.

constant m exists, positive or zero, such that $r(x') \geq r(x)$ when $x' \geq x + m$. A necessary and sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} u_n$ is the convergence of the integral

$$\int_0^{\infty} e^{\int_0^x \log r(x) dx} dx.$$

THEOREM III. Given the sequence $\{u_n\}$. Let $r(x)$ be a function with an integrable derivative $r'(x)$, such that $r(n) = r_n = u_{n+1}/u_n$. If

$$0 < A \leq r(x) \leq B,$$

and the integral

$$\int_0^{\infty} |r'(x)| dx$$

converges, a necessary and sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} u_n$ is the convergence of the integral

$$\int_0^{\infty} e^{\int_0^x \log r(x) dx} dx.$$

THEOREM IV. Given the sequence $\{u_n\}$. Let $r_n = u_{n+1}/u_n$, and let $r(x)$ be a positive, integrable function satisfying the preliminary conditions of one of the Theorems II or III, and the further condition that

$$|r(x) - 1| < a < 1, \quad \mu_1 < x.$$

A sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} u_n$ is the convergence of the integral

$$\int_0^{\infty} e^{\int_0^x (r(x)-1) dx} dx.$$

This condition is also necessary for the convergence of the series if from a certain point on

$$|r(x) - 1| < \frac{k}{x},$$

where k is some constant.

3. The Second-Difference Test. Suppose that we are given a sequence $\{u_n\}$ and a function $D(x)$ satisfying the condition that

$$D(n) = D_n = u_n - u_{n+1}.$$

By a method similar to that used in the last section we are led to an approximate form

$$u(x) = C - \int_0^x D(x) dx$$

for a function $u(x)$ such that $u(n) = u_n$. The integral in the Maclaurin-Cauchy test will clearly be of most interest if

$$C = \int_0^{\infty} D(x) dx.$$

Assigning this value to C , we are led to hope that under certain restrictions the given series will converge or diverge together with the integral

$$\int_0^{\infty} \int_x^{\infty} D(x) dx dx.$$

We have the following theorem.

THEOREM V. *Given the sequence $\{u_n\}$ where $\lim_{n \rightarrow \infty} u_n = 0$, and a positive, integrable function $D(x)$ such that $D(n) = D_n = u_n - u_{n+1}$, and such that either*

(α) there is an integer μ , positive or zero, for which $D(x) \geq D(x')$, whenever $x' \geq x + \mu$; or

(β) there is a constant k for which $D(x)/D(x') < k$, whenever $|x - x'| \leq 1$; a necessary and sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} u_n$ is the convergence of the integral

$$\int_0^{\infty} \int_x^{\infty} D(x) dx dx.$$

Proof. In case (α)

$$(1) \quad D_{n+\mu+1} \leq \int_n^{n+1} D(x) dx \leq D_{n-\mu}, \quad n > \mu.$$

Now

$$(2) \quad \int_n^{n+1} \int_x^m D(x) dx dx = \int_n^{n+1} \left[\int_x^{n+1} + \int_{n+1}^{n+2} + \int_{n+2}^{n+3} + \cdots + \int_{m-1}^m D(x) dx \right] dx$$

where m is any integer such that $m > n + 1 > \mu + 1$, and where, as elsewhere in this paper, the bar over the signs of integration indicates that all the integrals beneath it have the same integrand. Then, by (1),

$$(3) \quad \int_n^{n+1} \int_x^m D(x) dx dx \leq D_{n-\mu} + D_{n-\mu+1} + D_{n-\mu+2} + \cdots + D_{m-\mu-1} \\ = u_{n-\mu} - u_{m-\mu}.$$

Likewise

$$(4) \quad \int_n^{n+1} \int_x^m D(x) dx dx \geq D_{n+\mu+2} + D_{n+\mu+3} + \cdots + D_{m+\mu} \\ = u_{n+\mu+2} - u_{m+\mu+1}.$$

Therefore, since $\lim_{m \rightarrow \infty} u_m = 0$,

$$\lim_{n \rightarrow \infty} \int_n^{n+1} \int_x^m D(x) dx dx$$

exists, and

$$(5) \quad u_{n+\mu+2} \leq \int_n^{n+1} \int_x^\infty D(x) dx dx \leq u_{n-\mu}.$$

In case (β)

$$(6) \quad \frac{1}{k} D_n < \int_n^{n+1} D(x) dx < k D_n.$$

Then, by (2),

$$(7) \quad \int_n^{n+1} \int_x^m D(x) dx dx < k(D_n + D_{n+1} + \cdots + D_{m-1}) = k u_n - k u_m.$$

Likewise

$$(8) \quad \int_n^{n+1} \int_x^m D(x) dx dx > \frac{1}{k} u_{n+1} - \frac{1}{k} u_m.$$

Therefore, since $\lim_{m=\infty} u_m = 0$,

$$\lim_{m=\infty} \int_n^{n+1} \int_x^m D(x) dx dx$$

exists, and

$$(9) \quad \frac{1}{k} u_{n+1} \leq \int_n^{n+1} \int_x^\infty D(x) dx dx \leq k u_n.$$

The theorem follows from a comparison of the two series $\sum_{n=0}^\infty u_n$ and $\sum_{n=0}^\infty \int_n^{n+1} \int_x^\infty D(x) dx dx$ by means of (5) and (9).

Example. Test for convergence the following double series

$$\sum_{n=2}^\infty \sum_{n=n}^\infty \frac{1}{n^2(\log n)^p} \left[1 + \frac{p}{\log n} \right].$$

This can be taken as a double series of which the $(n-1)$ -st row is the simple series

$$u_n = \sum_{n=n}^\infty \frac{1}{n^2(\log n)^p} \left[1 + \frac{p}{\log n} \right].$$

We see at once that this simple series converges so that u_n exists and $\lim_{n=\infty} u_n = 0$. Write

$$D_n = u_n - u_{n+1} = \frac{1}{n^2(\log n)^p} \left[1 + \frac{p}{\log n} \right].$$

The conditions of the theorem hold, and

$$\int_{x_0}^\infty \int_x^\infty D(x) dx dx = \int_{x_0}^\infty \int_x^\infty \frac{1}{x^2(\log x)^p} \left[1 + \frac{p}{\log x} \right] dx dx = \int_{x_0}^\infty \frac{dx}{x(\log x)^p}.$$

Therefore the double series converges if $p > 1$, and diverges if $p \leq 1$.

This method involves essentially the same work as that involved in testing the series by the generalization to double series of the Maclaurin-Cauchy integral test. The present method has the slight advantage that only one variable of integration is used.

4. The Double-Ratio. Suppose that we are given the series

$$u_0 + u_1 + u_2 + \dots$$

We may call R_n the *double-ratio* of the series if

$$R_n = r_{n+1}/r_n = u_{n+2} \cdot u_n / (u_{n+1})^2.$$

Now suppose that $R(x)$ is a function such that $R(n) = R_n$. Then if $r(x)$ is properly defined in some initial interval the difference equation

$$\log R(x) = \log r(x+1) - \log r(x)$$

defines a function $r(x)$ having the property that $r(n) = r_n$. By the mean-value theorem we are led to the approximation

$$\log R(x) = r'(x)/r(x),$$

or

$$\log r(x) = \int_0^x \log R(x) dx + C.$$

Let us substitute this form in place of $\log r(x)$ in the test of § 2. We get the integral

$$\int_0^\infty e^{\int_0^x \left(\int_0^x \log R(x) dx + C \right) dx} dx.$$

The interesting case is that in which

$$C = - \int_0^\infty \log R(x) dx.$$

The following test is then suggested.

THEOREM VI. *Given the sequence $\{u_n\}$. Let $r_n = u_{n+1}/u_n$ and $R_n = r_{n+1}/r_n = u_{n+2} \cdot u_n / (u_{n+1})^2$. If $\lim_{n \rightarrow \infty} r_n = 1$, and if $R(x)$ is a function such that $R(n) = R_n$, and such that $R(x) \geq R(x')$ when $x' > x$, a necessary and sufficient condition for the convergence of the series $\sum_{n=0}^\infty u_n$ is the convergence of the integral*

$$\int_0^\infty e^{-\int_0^x \int_x^\infty \log R(x) dx dx} dx.$$

Proof. By definition

$$R_n \cdot R_{n+1} \cdot \dots \cdot R_m = \frac{1}{r_n} \cdot r_{m+1}, \quad m > n.$$

Then since $\lim_{m \rightarrow \infty} r_m = 1$, the infinite product $\prod_{k=n}^{\infty} R_k$ converges to the value $1/r_n$.

Let m and n be integers such that $0 \leq n < m$. Then since $R(x) \geq 1$ we have for $n \leq b \leq n+1$

$$(1) \quad \int_n^b \int_x^m \log R(x) dx dx \leq \int_n^b \int_n^m \log R(x) dx dx \\ \leq (b-n) \log (R_n R_{n+1} \cdots R_{m-1}).$$

Also

$$(2) \quad \int_n^b \int_x^m \log R(x) dx dx \geq \int_n^b \int_{n+1}^m \log R(x) dx dx \\ \geq (b-n) \log (R_{n+2} R_{n+3} \cdots R_m).$$

Therefore, since $\prod_{k=n}^{\infty} R_k = 1/r_n$,

$$\lim_{m \rightarrow \infty} \int_n^b \int_x^m \log R(x) dx dx$$

exists, and

$$(3) \quad \int_n^{n+1} \int_x^{\infty} \log R(x) dx dx \leq \log (1/r_n) = -\log r_n,$$

and

$$(4) \quad \int_n^{n+1} \int_x^{\infty} \log R(x) dx dx \geq \log (1/r_{n+2}) = -\log r_{n+2}.$$

Now

$$\int_n^{n+1} e^{-\int_0^x \int_x^{\infty} \log R(x) dx dx} dx \\ = \int_n^{n+1} e^{-\left[\int_0^1 + \int_1^2 + \int_2^3 + \cdots + \int_{n-1}^n + \int_n^x \left(\int_x^{\infty} \log R(x) dx \right) \right]} dx.$$

Consequently, by (4),

$$(5) \quad \int_n^{n+1} e^{-\int_0^x \int_x^{\infty} \log R(x) dx dx} dx \leq \int_n^{n+1} e^{\log (r_2 \cdot r_3 \cdots r_{n+1})} dx \\ = r_2 \cdot r_3 \cdots r_{n+1} = \frac{u_{n+2}}{u_2}.$$

And by (3)

$$(6) \quad \int_n^{n+1} e^{-\int_0^x \int_x^{\infty} \log R(x) dx dx} dx \geq \int_n^{n+1} e^{\log (r_0 \cdot r_1 \cdots r_n)} dx = r_0 \cdot r_1 \cdots r_n = \frac{u_{n+1}}{u_0}.$$

Since the integral given in the theorem does not oscillate, but converges or diverges with the series

$$\sum_{n=0}^{\infty} \int_n^{n+1} e^{-\int_0^x \int_x^{\infty} \log R(x) dx dx} dx,$$

a comparison of this series with the given series $\sum_{n=0}^{\infty} u_n$ by means of (5) and (6) establishes the theorem.

The theorem is easily generalized to include functions $R(x)$ which satisfy conditions similar to the conditions imposed upon $r(x)$ in Theorems II and III. For the sake of brevity it seems better to omit complete statements of these extensions.

Example. Test for convergence the following series

$$1 + e^{-\alpha \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)} + e^{-\alpha \left(1 + \frac{2}{2^2} + \frac{2}{3^2} + \frac{2}{4^2} + \cdots\right)} \\ + e^{-\alpha \left(1 + \frac{2}{2^2} + \frac{3}{3^2} + \frac{3}{4^2} + \frac{3}{5^2} + \cdots\right)} + \dots$$

Here

$$u_n = e^{-\alpha \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{n-1}{n^2} + \frac{(n-1)}{(n+1)^2} + \frac{(n-1)}{(n+2)^2} + \cdots\right)}, \\ r_n = e^{-\alpha \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots\right)}, \\ R_n = e^{\alpha/n^2}, \quad \text{and} \quad R(x) = e^{\alpha/x^2}.$$

Since $R(x)$ decreases monotonically for $\alpha > 0$, and since $\lim_{n \rightarrow \infty} r_n = 1$, the conditions for the theorem apply. Then

$$-\int_1^x \int_x^{\infty} \log R(x) dx dx = -\int_1^x \int_x^{\infty} \frac{\alpha}{x^2} dx = -\int_1^x \frac{\alpha}{x} dx = -\alpha \log x.$$

Therefore

$$\int_1^{\infty} e^{-\int_1^x \int_x^{\infty} \log R(x) dx dx} dx = \int_1^{\infty} \frac{dx}{x^{\alpha}}.$$

Consequently the series converges if $\alpha > 1$, and diverges if $\alpha \leq 1$.

By the mean-value theorem

$$(7) \quad \frac{\log R(x+h) - \log R(x)}{R(x+h) - R(x)} = \frac{R'(\xi)/R(\xi)}{R'(\xi)} = \frac{1}{R(\xi)},$$

where $x < \xi < x+h$. Under the conditions of the last theorem, $R(x) \geq 1$, and $\lim_{x \rightarrow \infty} R(x) = 1$. Let h become infinite. Then from (7)

$$(8) \quad \log R(x) = [R(x) - 1] \frac{1}{R(\xi)},$$

where $x < \xi$. Now $1/R(x) \leq 1$, but if k is any constant less than unity, there is a constant m such that $1/R(x) > k$ when $x > m$. Therefore

$$(9) \quad \log R(x) \leq R(x) - 1,$$

and

$$(10) \quad \log R(x) > k[R(x) - 1],$$

when $k < 1$ and $x > m$. Then the integral

$$\int_0^\infty e^{-\int_0^x \int_x^\infty \log R(x) dx dx} dx$$

diverges if the integral

$$(A) \quad \int_0^\infty e^{\int_0^x \int_x^\infty [1-R(x)] dx dx} dx$$

diverges, and it converges if the integral

$$(B) \quad \int_0^\infty e^{k \int_0^x \int_x^\infty [1-R(x)] dx dx} dx$$

converges. So a sufficient condition for the divergence of the series $u_0 + u_1 + u_2 + \dots$ is the divergence of the integral (A), and a sufficient condition for the convergence of the series is the convergence of the integral (B), if $0 < k < 1$.

In case neither of these conditions holds, so that the integral (A) converges but the integral (B) diverges, let us assume that there is a constant λ such that

$$(11) \quad 1 - R(x) > -\frac{\lambda}{x^2}.$$

Under the conditions of convergence and divergence just stated this assumption is commonly true.

We can write

$$(12) \quad \log R(x) = (R(x) - 1) - \frac{1}{2}(R(x) - 1)^2 + \frac{1}{3}(R(x) - 1)^3 - \dots$$

This series converges when x exceeds some fixed value. By assumption (11) we have

$$1 - R(x) > -\frac{\lambda}{x^2}, \quad (1 - R(x))^2 < \frac{\lambda^2}{x^4}, \quad (1 - R(x))^3 > -\frac{\lambda^3}{x^6}, \quad \text{etc.}$$

Then since $(1 - R(x)) < 0$,

$$-\log R(x) < (1 - R(x)) + \frac{\lambda^2}{2x^4} + \frac{\lambda^4}{4x^8} + \frac{\lambda^6}{6x^{12}} + \dots$$

And if $x > x_0 > \lambda$,

$$-\log R(x) < (1 - R(x)) + \frac{1}{x^4} \left[\frac{\lambda^2}{2} + \frac{\lambda^4}{4x_0^4} + \frac{\lambda^6}{6x_0^8} + \dots \right] = (1 - R(x)) + \frac{k}{x^4},$$

where k is a positive constant. Consequently

$$-\int_x^\infty \log R(x) dx < \int_x^\infty (1 - R(x)) dx + \frac{k}{3x^3}.$$

Integrating again between the limits x_0 and x , we have

$$\begin{aligned} - \int_{x_0}^x \int_x^{\infty} \log R(x) dx dx &< \int_{x_0}^x \int_x^{\infty} (1 - R(x)) dx dx + \frac{k}{6x_0^2} - \frac{k}{6x^2} \\ &< \int_{x_0}^x \int_x^{\infty} (1 - R(x)) dx dx + \frac{k}{6x_0^2}. \end{aligned}$$

It follows that if the integral

$$\int_{x_0}^{\infty} e^{\int_{x_0}^x \int_x^{\infty} (1 - R(x)) dx dx} dx$$

converges, so also will the integral

$$\int_{x_0}^{\infty} e^{-\int_{x_0}^x \int_x^{\infty} \log R(x) dx dx} dx.$$

We may therefore state the following theorem.

THEOREM VII. *Given the sequence $\{u_n\}$ and a differentiable function $R(x)$ satisfying the preliminary conditions of Theorem VI. If k is any constant less than unity, a sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} u_n$ is the convergence of the integral*

$$\int_0^{\infty} e^{k \int_0^x \int_x^{\infty} [1 - R(x)] dx dx} dx;$$

a necessary condition for the convergence of the series is the convergence of the integral

$$\int_0^{\infty} e^{\int_0^x \int_x^{\infty} [1 - R(x)] dx dx} dx,$$

and this condition is also sufficient if

$$(1 - R(x)) > -\frac{\lambda}{x^2}$$

where λ is some constant.

Example. Test for convergence the series

$$\begin{aligned} \sum_{n=\mu}^{\infty} \frac{1}{(1 + \alpha) \left(1 + \frac{\alpha}{2^2}\right)^2 \left(1 + \frac{\alpha}{3^2}\right)^3 \cdots \left(1 + \frac{\alpha}{(n-1)^2}\right)^{n-1}} \\ \times \left(1 + \frac{\alpha}{n^2}\right)^{n-1} \left(1 + \frac{\alpha}{(n+1)^2}\right)^{n-1} \cdots \end{aligned}$$

For this series

$$r_n = \frac{1}{\left(1 + \frac{\alpha}{n^2}\right) \left(1 + \frac{\alpha}{(n+1)^2}\right) \left(1 + \frac{\alpha}{(n+2)^2}\right) \cdots},$$

and $R_n = 1 + (\alpha/n^2)$, so that $R(x) = 1 + (\alpha/x^2)$. The conditions for the theorem are clearly satisfied.

$$\int_1^x \int_x^\infty (1 - R(x)) dx dx = \int_1^x \int_x^\infty -\frac{\alpha}{x^2} dx dx = -\int_1^x \frac{\alpha}{x} dx = -\alpha \log x.$$

Therefore

$$\int_1^\infty e^{\int_1^x \int_x^\infty [1 - R(x)] dx dx} dx = \int_1^\infty \frac{dx}{x^\alpha},$$

so that the given series converges if $\alpha > 1$, and diverges if $\alpha \leq 1$.

A series for which the double-ratio function is known may be written in the normal form

$$a + b + \frac{b^2}{a} a_0 + \frac{b^3}{a^2} a_0^2 a_1 + \frac{b^4}{a^3} a_0^3 a_1^2 a_2 + \cdots,$$

where a_n is the value $a(n)$ of a known function $a(x)$. It is to such series that the theorems of this section apply.

5. The Difference of Ratios. Given a sequence $\{u_n\}$. Let the ratio be $r_n = u_{n+1}/u_n$ and the *difference of ratios* be

$$\rho_n = r_{n+1} - r_n = \frac{u_{n+2}}{u_{n+1}} - \frac{u_{n+1}}{u_n}.$$

If $\rho(x)$ is a function such that $\rho(n) = \rho_n$, and if $r(x)$ is properly defined in some initial interval, the difference equation

$$\rho(x) = r(x+1) - r(x)$$

defines the function $r(x)$ so that $r(n) = r_n$. Then $\rho(x) = r'(\xi)$, $x < \xi < x+1$, and we expect that if suitable restrictions are placed upon $\rho(x)$, a relatively small error will be made if, for large values of x , we set $\rho(x) = r'(x)$ or

$$r(x) = \int_0^x \rho(x) dx + C.$$

The integral of § 2 becomes

$$\int_0^\infty e^{\int_0^x \log \left(C + \int_0^x \rho(x) dx \right) dx} dx.$$

In the interesting case we shall have

$$\lim_{x \rightarrow \infty} \int_0^x \rho(x) dx + C = 1, \quad \text{or} \quad C + \int_0^x \rho(x) dx = 1 - \int_x^\infty \rho(x) dx.$$

This suggests the following theorem.

THEOREM VIII. *Given the sequence $\{u_n\}$. Let $r_n = u_{n+1}/u_n$, and $\rho_n = r_{n+1} - r_n$. Suppose that $r_n < 1$ and that $\lim_{n \rightarrow \infty} r_n = 1$. Let $\rho(x)$ be a*

function such that

$$(1) \quad \rho(n) = \rho_n,$$

$$(2) \quad 0 < \rho(x) < 1,$$

$$(3) \quad \rho(x) \text{ decreases monotonically when } x \text{ increases.}$$

A necessary and sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} u_n$ is the convergence of the integral

$$\int_0^{\infty} e^{\int_0^x \log(1 - \int_x^{\infty} \rho(x) dx)} dx.$$

Proof.

$$(1) \quad \rho_{n+1} \equiv \int_n^{n+1} \rho(x) dx \equiv \rho_n.$$

If n and m are integers such that $n < m$, and if $n \leq b \leq n+1$, we have

$$\begin{aligned} (2) \quad & \int_n^b \log \left(1 - \int_x^m \rho(x) dx \right) dx \\ & \equiv (b-n) \log \left[1 - \sqrt[n+1]{\int_{n+1}^{n+2} - \int_{n+2}^{n+3} - \cdots - \int_{m-1}^m \rho(x) dx} \right] \\ & \equiv (b-n) \log [1 - \rho_{n+2} - \rho_{n+3} - \cdots - \rho_m] \\ & = (b-n) \log (1 - r_{m+1} + r_{n+2}). \end{aligned}$$

Since $r_k < 1$, this logarithm is defined for all values of n and m . Likewise

$$\begin{aligned} (3) \quad & \int_n^b \log \left(1 - \int_x^m \rho(x) dx \right) dx \\ & \equiv (b-n) \log \left[1 - \sqrt[n+1]{\int_{n+1}^{n+2} - \int_{n+2}^{n+3} - \cdots - \int_{m-1}^m \rho(x) dx} \right] \\ & \equiv (b-n) \log [1 - \rho_n - \rho_{n+1} - \cdots - \rho_{m-1}] \\ & = (b-n) \log (1 - r_m + r_n). \end{aligned}$$

Therefore, since $\lim_{m \rightarrow \infty} r_m = 1$,

$$\lim_{m \rightarrow \infty} \int_n^b \log \left(1 - \int_x^m \rho(x) dx \right) dx$$

exists, $n \leq b \leq n+1$, and

$$(4) \quad \log r_n \equiv \int_n^{n+1} \log \left(1 - \int_x^{\infty} \rho(x) dx \right) dx \equiv \log r_{n+2}.$$

Now

$$\begin{aligned} \int_n^{n+1} e^{\int_0^x \log \left(1 - \int_x^\infty \rho(x) dx\right) dx} dx \\ = \int_n^{n+1} e^{\sqrt{\int_0^1 + \int_1^2 + \dots + \int_{n-1}^n + \int_n^x} \log \left(1 - \int_x^\infty \rho(x) dx\right) dx} dx. \end{aligned}$$

Consequently, by (4),

$$(5) \quad \int_n^{n+1} e^{\int_0^x \log \left(1 - \int_x^\infty \rho(x) dx\right) dx} dx < e^{\log(r_2 \cdot r_3 \cdot \dots \cdot r_{n+1})} = r_2 \cdot r_3 \cdot \dots \cdot r_{n+1} = \frac{u_{n+2}}{u_2}.$$

Also

$$(6) \quad \int_n^{n+1} e^{\int_0^x \log \left(1 - \int_x^\infty \rho(x) dx\right) dx} dx > e^{\log(r_0 \cdot r_1 \cdot \dots \cdot r_n)} = r_0 \cdot r_1 \cdot \dots \cdot r_n = \frac{u_{n+1}}{u_0}.$$

The theorem follows at once from a comparison of the two series $\sum_{n=0}^{\infty} u_n$

$$\text{and } \sum_{n=0}^{\infty} \left[\int_n^{n+1} e^{\int_0^x \log \left(1 - \int_x^\infty \rho(x) dx\right) dx} dx \right].$$

We have the expansion

$$\begin{aligned} (7) \quad & \log \left(1 - \int_x^\infty \rho(x) dx\right) \\ &= - \int_x^\infty \rho(x) dx - \frac{1}{2} \left[\int_x^\infty \rho(x) dx \right]^2 - \frac{1}{3} \left[\int_x^\infty \rho(x) dx \right]^3 - \dots. \end{aligned}$$

Hence $\log \left(1 - \int_x^\infty \rho(x) dx\right) < - \int_x^\infty \rho(x) dx$. Therefore if the integral

$$\int_0^\infty e^{-\int_0^x \int_x^\infty \rho(x) dx} dx$$

converges, so also will the integral

$$\int_0^\infty e^{\int_0^x \log \left(1 - \int_x^\infty \rho(x) dx\right) dx} dx.$$

In case the first of these integrals diverges, it will often happen that

$$\rho(x) < \frac{\lambda}{x^2},$$

λ being a constant. Let us assume that this condition holds. Then

from (7) we have for $x > x_0 > \lambda$,

$$\begin{aligned}
 \log \left(1 - \int_x^\infty \rho(x) dx \right) &> - \int_x^\infty \rho(x) dx - \frac{1}{2} \left[\int_x^\infty \frac{\lambda}{x^2} dx \right]^2 \\
 &- \frac{1}{3} \left[\int_x^\infty \frac{\lambda}{x^2} dx \right]^3 - \cdots > - \int_x^\infty \rho(x) dx \\
 &- \frac{1}{x^2} \left(\frac{\lambda^2}{2} + \frac{\lambda^3}{3x_0} + \frac{\lambda^4}{4x_0^2} - \cdots \right) = - \int_x^\infty \rho(x) dx - \frac{k}{x^2},
 \end{aligned}
 \tag{8}$$

where k is a positive constant. Consequently

$$\begin{aligned}
 \int_\mu^\infty \log \left(1 - \int_x^\infty \rho(x) dx \right) dx &> - \int_\mu^\infty \int_x^\infty \rho(x) dx dx - \int_\mu^\infty \frac{k}{x^2} dx \\
 &= \int_\mu^\infty \int_x^\infty \rho(x) dx dx + \frac{k}{x} - \frac{k}{\mu} > - \frac{k}{\mu} - \int_\mu^\infty \int_x^\infty \rho(x) dx dx.
 \end{aligned}
 \tag{9}$$

It follows that

$$\int_\mu^\infty e^{\int_\mu^x \log \left(1 - \int_x^\infty \rho(x) dx \right) dx} dx > C \int_\mu^\infty e^{-\int_\mu^x \int_x^\infty \rho(x) dx dx} dx.$$

Therefore if the integral

$$\int_\mu^\infty e^{-\int_\mu^x \int_x^\infty \rho(x) dx dx} dx$$

diverges, the integral

$$\int_\mu^\infty e^{\int_\mu^x \log \left(1 - \int_x^\infty \rho(x) dx \right) dx} dx$$

also diverges. We then have the following test.

THEOREM IX. *Given the sequence $\{u_n\}$. Let $r_n = u_{n+1}/u_n < 1$, and $\lim_{n \rightarrow \infty} r_n = 1$. Let $\rho(x)$ be a function such that $\rho(n) = \rho_n = r_{n+1} - r_n$, $0 < \rho(x) < 1$, and $\rho(x)$ decreases monotonically. The convergence of the integral*

$$\int_0^\infty e^{-\int_0^x \int_x^\infty \rho(x) dx dx} dx$$

is sufficient for the convergence of the series $\sum_{n=0}^\infty u_n$. Moreover if there is a constant λ such that from a certain point on $\rho(x) < \lambda/x^2$, the convergence of the integral is necessary for the convergence of the given series.

Example. Test for convergence the series $\sum_{n=1}^\infty u_n$ where

$$u_n = \left[1 - \sum_{k=\mu}^\infty \frac{\alpha}{k^2} \right] \left[1 - \sum_{k=\mu}^\infty \frac{\alpha}{(k+1)^2} \right] \cdots \left[1 - \sum_{k=\mu}^\infty \frac{\alpha}{(k+n-1)^2} \right]$$

μ being so large that $u_n > 0$ for all positive integral values of n . For this series

$$r_n = \frac{u_{n+1}}{u_n} = \left[1 - \sum_{k=\mu}^{\infty} \frac{\alpha}{(k+n)^2} \right].$$

Then $\lim_{n \rightarrow \infty} r_n = 1$. Also

$$\rho_n = \frac{\alpha}{(\mu + n)^2}, \quad \rho(x) = \frac{\alpha}{(\mu + x)^2}.$$

The conditions of the theorem are satisfied.

$$- \int_1^x \int_x^{\infty} \rho(x) dx dx = - \int_1^x \int_x^{\infty} \frac{\alpha}{(\mu + x)^2} dx dx = \int_1^x \frac{-\alpha}{\mu + x} dx = -\log(\mu + x)^{\alpha}.$$

Therefore

$$\int_1^{\infty} e^{-\int_1^x \int_x^{\infty} \rho(x) dx dx} dx = \int_1^{\infty} \frac{dx}{(\mu + x)^{\alpha}},$$

and the given series converges if $\alpha > 1$, and diverges if $\alpha \leq 1$.

6. **The Ratio of Differences.** Let us call

$$\delta_n = \frac{u_{n+2} - u_{n+1}}{u_{n+1} - u_n}$$

the *ratio of differences* for the series

$$u_0 + u_1 + u_2 + \dots$$

Suppose that $\delta(x)$ is a function such that $\delta(n) = \delta_n$. By the same sort of work as that given in the preceding sections, the difference equation in $u(x)$

$$\delta(x) = \frac{u(x+2) - u(x+1)}{u(x+1) - u(x)}$$

gives

$$\delta(x) = \frac{u'(\xi+1)}{u'(\xi)} = e^{u''(\eta)/u'(\eta)}, \quad x < \xi < \eta < x+2,$$

which suggests as a useful differential equation

$$\log \delta(x) = \frac{u''(x)}{u'(x)}.$$

Then, by the methods already used, the following test is suggested.

THEOREM X. Given the sequence $\{u_n\}$ where $\lim_{n \rightarrow \infty} u_n = 0$. Let

$$\delta_n = \frac{u_{n+2} - u_{n+1}}{u_{n+1} - u_n},$$

and let $\delta(x)$ be a function having the following properties.

- (1) $\delta(n) = \delta_n,$
- (2) $\delta(x) \leq 1,$
- (3) $\delta(x') \leq \delta(x), \quad x' < x.$

A necessary and sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} u_n$ is the convergence of the integral

$$\int_0^{\infty} \int_x^{\infty} e^{\int_0^x \log \delta(x) dx} dx dx.$$

Proof.

$$(1) \quad \log \delta_n \leq \int_n^{n+1} \log \delta(x) dx \leq \log \delta_{n+1}.$$

$$(2) \quad \int_n^{n+1} e^{\int_0^x \log \delta(x) dx} dx = \int_n^{n+1} e^{\overbrace{\int_0^1 + \int_1^2 + \dots + \int_{n-1}^n}^{\int_0^x} \log \delta(x) dx} dx \\ \leq e^{\log(\delta_1 \cdot \delta_2 \cdot \dots \cdot \delta_n)} = \delta_1 \cdot \delta_2 \cdot \dots \cdot \delta_n, \quad n \geq \mu.$$

Similarly

$$(3) \quad \int_n^{n+1} e^{\int_0^x \log \delta(x) dx} dx \geq \delta_0 \cdot \delta_1 \cdot \dots \cdot \delta_n.$$

Now

$$\int_n^{n+1} \int_x^m e^{\int_0^x \log \delta(x) dx} dx dx \\ = \int_n^{n+1} \left[\overbrace{\int_x^{n+1} + \int_{n+1}^{n+2} + \dots + \int_{m-1}^m}^{\int_x^m} e^{\int_0^x \log \delta(x) dx} \right] dx.$$

Therefore, by (2),

$$(4) \quad \int_n^{n+1} \int_x^m e^{\int_0^x \log \delta(x) dx} dx dx \leq [\delta_1 \cdot \delta_2 \cdot \dots \cdot \delta_n + \delta_1 \cdot \delta_2 \cdot \dots \cdot \delta_{n+1} + \dots \\ + \delta_1 \cdot \delta_2 \cdot \dots \cdot \delta_{m-1}],$$

and by (3),

$$(5) \quad \int_n^{n+1} \int_x^m e^{\int_0^x \log \delta(x) dx} dx dx \geq [\delta_0 \cdot \delta_1 \cdot \dots \cdot \delta_{n+1} + \delta_0 \cdot \delta_1 \cdot \dots \cdot \delta_{n+2} + \dots \\ + \delta_0 \cdot \delta_1 \cdot \dots \cdot \delta_{m-1}].$$

Moreover

$$\delta_0 \cdot \delta_1 \cdot \dots \cdot \delta_n = \frac{u_{n+2} - u_{n+1}}{u_1 - u_0}.$$

This division by $u_1 - u_0$ is justified by the conditions imposed on $\delta(x)$. It follows that

$$\begin{aligned}
& \lim_{m=\infty} [\delta_0 \cdot \delta_1 \cdot \dots \cdot \delta_n + \delta_0 \cdot \delta_1 \cdot \dots \cdot \delta_{n+1} + \dots + \delta_0 \cdot \delta_1 \cdot \dots \cdot \delta_{m-1}] \\
&= \lim_{m=\infty} \frac{1}{u_1 - u_0} [(u_{n+2} - u_{n+1}) + (u_{n+3} - u_{n+2}) + \dots + (u_{m+1} - u_m)] \\
&= \lim_{m=\infty} \frac{u_{m+1} - u_{n+1}}{u_1 - u_0} = \frac{u_{n+1}}{u_0 - u_1}.
\end{aligned}$$

Consequently, by (4) and (5),

$$(7) \quad \int_n^{n+1} \int_x^\infty e^{\int_0^x \log \delta(x) dx} dx dx \leq \frac{u_{n+1}}{u_1 - u_2},$$

and

$$(8) \quad \int_n^{n+1} \int_x^\infty e^{\int_0^x \log \delta(x) dx} dx dx \geq \frac{u_{n+2}}{u_0 - u_1}.$$

The theorem then follows from a comparison of the two series $\sum_{n=0}^{\infty} u_n$ and

$$\sum_{n=0}^{\infty} \int_n^{n+1} \int_x^\infty e^{\int_0^x \log \delta(x) dx} dx dx.$$

Example. Test the double series

$$\sum_{n=2}^{\infty} \sum_{l=n-1}^{\infty} e^{-\sum_{k=1}^l \frac{p}{k}}.$$

We can write this as $\sum_{n=2}^{\infty} u_n$ where

$$u_n = \sum_{l=n-1}^{\infty} e^{-\sum_{k=1}^l \frac{p}{k}}.$$

First let us determine under what conditions the series defining u_n converges.

We can write

$$u_n = C(1 + e^{-\frac{p}{n}} + e^{-\frac{p}{n} - \frac{p}{n+1}} + e^{-\frac{p}{n} - \frac{p}{n+1} - \frac{p}{n+2}} + \dots) = C + C \sum_{k=0}^{\infty} v_k,$$

where $v_k = e^{-\frac{p}{n} - \frac{p}{n+1} - \dots - \frac{p}{n+k}}$. For the series $\sum_{k=0}^{\infty} v_k$, the ratio is

$$r_k = v_{k+1}/v_k = e^{-\frac{p}{k+n+1}}, \quad \text{and} \quad r(x) = e^{-\frac{p}{n+x+1}}.$$

We can apply Theorem II. We get

$$(9) \quad \int_0^x e^{\int_\mu^x \log r(x) dx} dx = \int_0^x e^{\int_\mu^x \frac{p}{x+n+1} dx} dx = C_1 \int_0^x \frac{dx}{(x+n+1)^p},$$

and the series $\sum_{k=0}^{\infty} v_k$ converges, and u_n is defined, if and only if $p > 1$.

If $p > 1$ we see also that $\lim_{n=\infty} u_n = 0$, for u_n is the remainder after $(n-2)$

terms of the convergent series that defines u_2 . We have

$$\delta_n = \frac{u_{n+2} - u_{n+1}}{u_{n+1} - u_n} = \frac{e^{\sum_{k=1}^n \frac{p}{k}}}{e^{-\sum_{k=1}^{n-1} \frac{p}{k}}} = e^{-\frac{p}{n}},$$

$$\delta(x) = e^{-\frac{p}{x}}.$$

We may apply Theorem X. As in (9), we have

$$(10) \quad \int_x^\infty e^{\int_0^x \log \delta(x) dx} dx = C \int_x^\infty \frac{dx}{x^p} = \frac{C}{(p-1)x^{p-1}},$$

$$(11) \quad \int_0^\infty \int_x^\infty e^{\int_0^x \log \delta(x) dx} dx dx = \frac{C}{p-1} \int_0^\infty \frac{dx}{x^{p-1}}.$$

Therefore the given series converges if $p > 2$, and diverges if $p \leq 2$. This same series is conveniently tested by means of a generalization to multiple series of the test of Theorem II.

7. Other Tests. The following test is immediately suggested and established by the methods already used.

THEOREM XI. *Given the sequence $\{u_n\}$. Let $D_n = u_n - u_{n+1}$, and $t_n = D_n - D_{n+1}$. If $\lim_{n \rightarrow \infty} u_n = 0$, and $t(x)$ be a positive function such that $t(n) = t_n$, and $t(x') \geq t(x)$, when $x > x'$, then a necessary and sufficient condition for the convergence of the series $\sum_{n=0}^\infty u_n$ is the convergence of the integral*

$$\int_0^\infty \int_x^\infty \int_x^\infty t(x) dx dx dx.$$

The difference equation of the fourth order for $S(x)$ that leads to the following test, as well as the proof of the test are readily suggested by the methods used elsewhere in this paper.

THEOREM XII. *Given the sequence $\{u_n\}$. Let $r_n = u_{n+1}/u_n$, $R_n = r_{n+1}/r_n$, and*

$$\rho_n = R_{n+1}/R_n = \frac{u_{n+3} \cdot u_{n+1}^3}{u_n \cdot u_{n+2}^3}.$$

If $\lim_{n \rightarrow \infty} r_n = 1$, and $\rho(x)$ be a positive function such that $\rho(n) = \rho_n$, and $\rho(x) \leq 1$, and $\rho(x') \geq \rho(x)$ when $x' > x$, then a necessary and sufficient condition for the convergence of the series $\sum_{n=0}^\infty u_n$ is the convergence of the integral

$$\int_0^\infty e^{\int_0^x \int_x^\infty \int_x^\infty \log \rho(x) dx dx dx} dx.$$

8. **Conclusion.** From the examples given, the general method is revealed by which one can obtain an infinite sequence of tests, of which those given above are the simplest and most useful.

Tests for multiple series are readily suggested by analogy with the foregoing tests for simple series. A test for multiple series analogous to the test of Theorem II is given in the author's paper to which reference has already been made. As is shown by some of the examples of this paper some of the foregoing tests can be used directly to examine certain types of multiple series.

The theorems of this paper have been stated for series of constant terms. Many of the tests can readily be extended to series of functions, not only to test such series for convergence, but also to determine whether the convergence is uniform, uniform convergence of an integral implying uniform convergence of the corresponding series.

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